18.152 PROBLEM SET 3

due March 14th 9:30 am (Gradescope).

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Ω is a bounded open subset of $\mathbb{R}^2$ and smooth boundary. In addition, g is a smooth function defined on $\partial\Omega$. Assume that u is a smooth harmonic function defined on $\overline{\Omega}$ satisfying $u = g$ on $\partial\Omega$.*

(1) *Show that $|u(x)| \leq \max\limits_{\partial\Omega} |g(x)|$.*
(2) *Given a ball $B_r(x_0) \in \Omega$, establish an upper bound for $|\nabla u(x_0)|^2$ in terms g and r by using the cut-off function $\eta = (r^2 - |x - x_0|^2)_+$ and the cut-off function.*
(3) *Given a ball $B_r(x_0) \in \Omega$, establish an upper bound for $|\nabla u(x_0)|^2$ in terms g and r by using Poisson's formula.*

Hint for 1-(2): Consider $w = \eta^2|\nabla u|^2 + Ku^2$ for constant K. And find out a sufficiently large K to make w a subsolution.

Problem 2. *Ω is a bounded open subset of $\mathbb{R}^2$ and smooth boundary. In addition, g is a smooth function defined on $\partial\Omega$. Assume that u is a smooth function defined on $\overline{\Omega}$ satisfying $u = g$ on $\partial\Omega$ and the following equation on Ω.*

$$(1 + u_x)u_{xx} + \frac{u_{yy}}{1 + x^2 + y^2} = 0. \tag{1}$$

Show that $|u(x)| \leq \max_{\partial\Omega} |g(x)|$.

Hint: We can consider $u_{xx} = b_x$, and apply the maximum principle.

Problem 3. *Show that the Green function to a smooth bounded domain Ω is symmetric. Namely,*

$$G(x, y) = G(y, x). \tag{2}$$

Hint: Given $x \neq y$, choose $\epsilon < |x - y|$. Next, define $u(z) = G(x, z)$ and $v(z) = G(y, z)$, and apply the Green's identity over the domain $\Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y))$. By passing $\epsilon \to 0$, show $u(y) = v(x)$.

Problem 4 (Liouville theorem)**.** *Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a non-negative harmonic function. Show that u is a constant function.*

Apply the Harnack's inequality to a large ball $B_R(0)$. Pass R to $+\infty$.

Problem 5 (Bonus)**.** *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a rectangle, and* $G(x, y)$ *is the corresponding Green function. Show that given smooth* f, g *the Green's representation*

$$u(x) = -\int_\Omega f(y)G(x, y)dy - \int_{\partial\Omega} g(y)G_\nu(x, y)dy \tag{3}$$

solves the Dirichlet problem $\Delta u = f$ *in* Ω *and* $u = g$ *on* $\partial\Omega$*.*

Hint: Suppose that the sides of Ω are parallel to the x, y-axis. Observe that the Laplacian of boundary integral is zero. Regarding the integral over Ω, split G into $\Phi + \varphi$. Show that $\Delta\varphi(x, y) = 0$ by using the result of problem 3. For the Φ, introduce a new variable $z = y - x$, and the corresponding domain $\Omega_x = \{y - x : y \in \Omega\}$. Split the integral region Ω_x into a small ball $B_\epsilon(0)$ and its outside. *Consider the derivative of the integral over* Ω_x*.*